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Verifying Cyber-Physical Systems

Chapter 7 – Verifying Invariants

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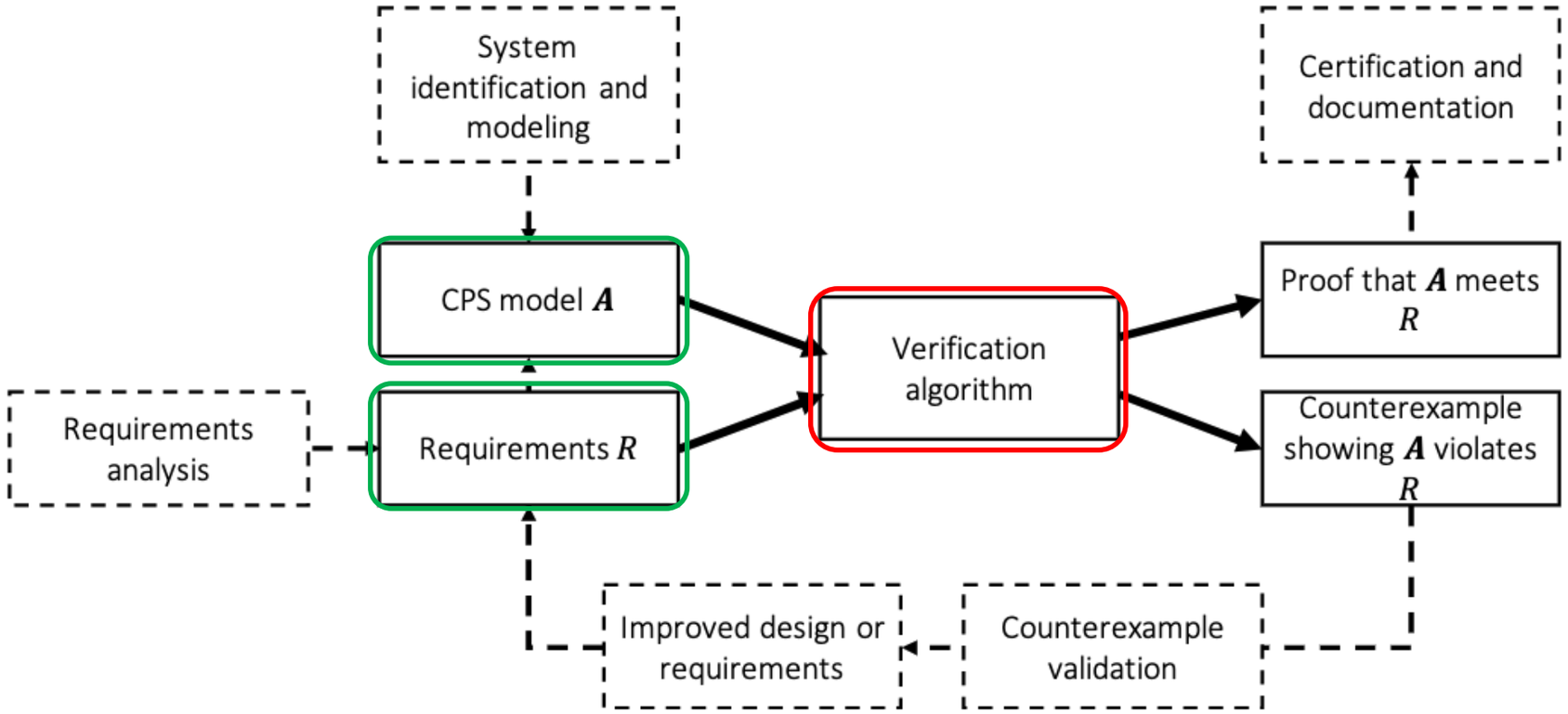
System Design Ecosystem



What is an invariant?

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- An invariant I for a system $\mathcal{A}$ is a requirement that **always holds** (I is never violated in any execution of $\mathcal{A}$).
- **Formally**: for a hybrid automata $\mathcal{A} =< V, \Theta, A, \mathcal{D}, \mathcal{T} >$, an invariant I is a subset of the state-space V such that all the reachable states of $\mathcal{A}$ are contained in I.

Theorem 7.1. *Given an HIOA $\mathcal{A}$, if a set of states $I \subseteq val(X)$ satisfies the following:*

(a) (Start condition) For any starting state $\mathbf{x} \in \Theta$, $\mathbf{x} \in I$, and

(b) (Transition closure) for any action $a \in A$, if $\mathbf{x} \xrightarrow{a} \mathbf{x}'$ and $\mathbf{x} \in I$ then $\mathbf{x}' \in I$, and

(c) (Trajectory closure) for any trajectory $\tau \in \mathcal{T}$, if τ.fstate $\in I$ then τ.lstate $\in I$,

then I is an inductive invariant of $\mathcal{A}$.

Historical perspective on invariants

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- The idea of Invariants dates to the classical program analysis technique called **Floyd-Hoare style verification** (Floyd, 1967).
- **Hoare logic/ Floyd-Hoare logic/ Hoare rules** – provides a set of logical rules for rigorously deducing the correctness of automata or programs.
- **Hoare logic** is built on a notion called **Hoare triple** – describes how the execution of a statement changes the state of the automaton.
- Hoare logic **provide axioms and inference rules** that various programming language constructs can use to infer properties from the triples.

Proving invariants via inductive reasoning

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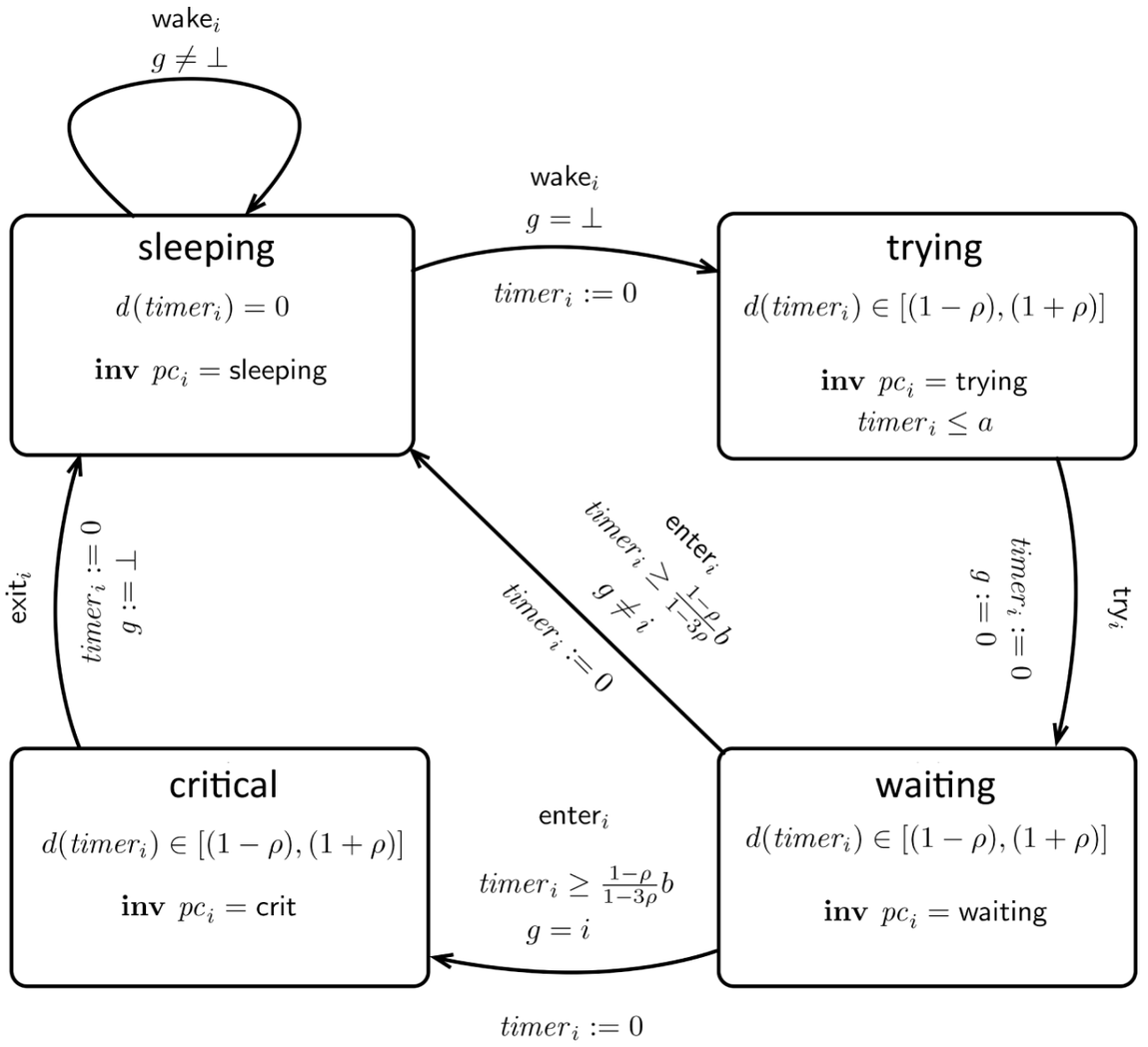
- Example: Fisher's mutual exclusion algorithm
- The algorithm allows n processes to coordinate and access a critical resource exclusively.
  - No two processes have simultaneous access, and all requesting processes eventually gain access
- **Our aim**: establish the correctness of a distributed mutual exclusion algorithm
- $g - single\ shared\ memory$
- $phases - sleeping, trying, waiting, and\ critical$
- $internal\ variables$: $pc_i -$ $program\ counter, and\ timer_i - stopwatch$

$internal\ actions$: $wake, try, enter, exit$



Analysis of Fisher's mutual exclusion

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- **Ultimate goal**: to establish the safety requirement for $\mathcal{A}$ that no two processes enter the critical section simultaneously.

$$\forall\, i, j \in [n], i \neq j \wedge pc_i = critical\ implies\ pc_j \neq critical$$

- **Approach**: establish a sequence of simpler invariants
- **Invariant 1**: $\forall\, i \in [n], if\ pc_i = sleeping,\ then\ timer_i = 0; otherwise\ timer_i \geq 0$
  - This simply states that for any process i, the timer is always nonnegative and stays at 0 when the process is sleeping
- **Invariant 2**: $\forall\, i, j \in [n],\ pc_i = waiting \wedge pc_j = trying \wedge g = i\ implies$

$$timer_i - timer_j < \frac{2\rho}{1-\rho} timer_i + b - a$$

  - This asserts the gap between the timer values of two processes that are attempting to access the critical section
- **Invariant 3**: $\forall\, i, j \in [n]\ pc_i = critical\ implies\ pc_j \neq trying \wedge g = i$
  - If process i is already in the critical section, then $g = i$, and no other process could be trying.

Barriers to automated invariant proving

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- Two barriers to automating the process of proving invariants
- **Barrier 1**: automating the strengthening step → requires an understanding of how the system works
- **Barrier 2**: need for explicit trajectories for checking the trajectory condition of Theorem 7.1
- Addressing barrier 2: proving invariants without solving ODEs
- **Main idea**: look at the boundary of the invariant set I, and prove that the trajectories cannot escape the boundary
- **Lemma**: I is an inductive invariant of an ODE $\dot{x} = f(x)$ if at every state x on the boundary of I, the vector $f(x)$ is pointing inward from the boundary $\frac{\partial F(x)}{\partial x} . f(x) \geq 0$.
- The above condition is called **sub-tangential condition**, and the boundary is called **barrier**.

Barrier certificates

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- They can be used to construct safety proofs relative to an unsafe set.
- Key idea: find a barrier function that separates the unsafe set from the reachset

Theorem 7.5. *Let $\mathcal{A} = (V, \Theta, A, \mathcal{D}, \mathcal{T})$ be a simple hybrid automaton with $V = X \cup loc$, set of locations $L = type(loc)$, and unsafe set $U \subseteq val(V)$. Suppose that there is a collection $\{B_\ell(\boldsymbol{x})\}$ of functions B_ℓ: $val(X) \to \mathbb{R}$ such that for all $\ell \in L$, B_ℓ is differentiable with respect to its argument, and it satisfies:*

(a) (Start condition) *For any starting state $\boldsymbol{v} \in \Theta_\ell$, $B_\ell(\boldsymbol{x}) \leq 0$.*

(b) (Unsafe condition) *For any unsafe state $\boldsymbol{v} \in U_\ell$, $B_\ell(\boldsymbol{x}) > 0$.*

(c) (Transition condition) *For any transition $(\mathbf{x}, \ell) \xrightarrow{a} (\mathbf{x}', \ell')$, $B_\ell(\mathbf{x}) \leq 0 \Rightarrow B'_\ell(\mathbf{x}') \leq 0$.*

(d) (Flow condition) *For any state $\boldsymbol{x} \in I_\ell$, on the boundary with $B_\ell(\boldsymbol{x}) = 0$, $\frac{\partial_\ell}{\partial \mathbf{x}} f_\ell(\mathbf{x}) \leq 0$.*

Then $\mathcal{A}$ is safe with respect to U (i.e., $\mathsf{Reach}_{\mathcal{A}} \cap U = \emptyset$).

Satisfiability and satisfiability modulo theories

- Satisfiability:
  - Given a propositional logic formula ∅ over a set of variables X, decide if there is a valuation that satisfies ∅.
- For example: given the propositional formula

$$\phi(x_1, x_2, x_3) := (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_1),$$

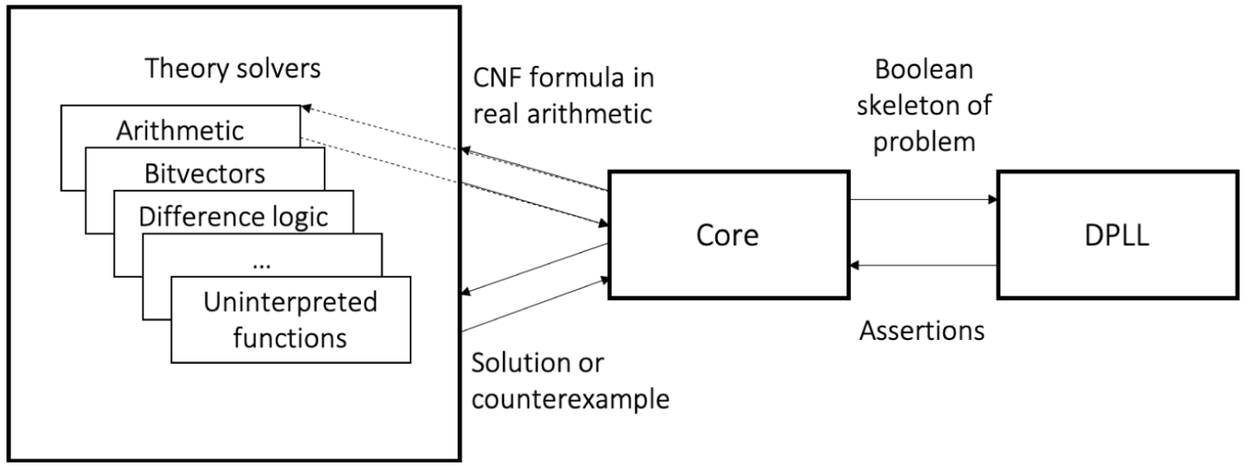
- The SAT problem decides whether $\exists\ x_1, x_2, x_3 : \phi(x_1, x_2, x_3).$
- Approaches to solving the SAT problem
  - Brute-force → check ∅ for each possible valuation $x \in val(X)$ of X ➔ requires $O(2^n)$ operations for a set of n variables
  - Davis Putnam Logeman and Loveland (DPLL) → backtracking algorithm that most solvers build on

Satisfiability modulo theory (SMT)

- SMT is a generalization of satisfiability in which some of the binary variables are replaced by predicates over nonbinary variables
- Example: $\phi_1(w, x, y, z) := (x - y = 5) \wedge (z - y \geq 2) \wedge (z - x > 2) \wedge (w - x = 2)$

  $\phi_2(x, y, z) := (3x^2 - 4y + 5z \leq 5) \wedge (-2x + 5z^3 \leq 7).$
- $\emptyset_1$ is an example of a predicate in **difference logic (DL),** and $\emptyset_2$ is a predicate in **real arithmetic**
- The satisfaction of predicates is defined in terms of the appropriate theories
- For example:
  - Linear inequalities over real variables are evaluated using the rules of the theory of linear real arithmetic
  - Predicates involving uninterpreted function symbols are evaluated using the rules of the theory of uninterpreted functions
  - These theories are called background theories for SMT
- An SMT solver uses theory solvers together with DPLL

Combine theory solvers with DPLL



Lazy approach to combine theory solvers with DPLL

1. Construct a Boolean abstraction $abs(\phi)$ of the given formula ϕ.
2. Use a SAT solver to find a satisfying solution x for $abs(\phi)$; if the SAT solver returns "unsatisfiable," then conclude that ϕ is unsatisfiable; otherwise, go to step 3.
3. Use a theory solver to check the satisfiability of $abs^{-1}(x)$ obtained from step 2.
4. If the theory solver returns "unsatisfiable," then a new constraint is added to $abs(\phi)$, and steps 1–4 are repeated.

Finding and learning invariants

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- The problem of finding adequately strong invariants for hybrid automata is a challenging one
- **Approach 1: Invariant generation with templates:**
  - Basic idea is to choose a template form of the invariants $I(p)$ and then solve constraints imposed by Theorem 7.1 to find the unknown coefficients p
  - One example can be using the template shape of boxes or hyperrectangles defined by parameters $l, u \in R^n$

$$Box(l,u) = \{v \in val(V) || l_i \leq v_i \leq u_i\}$$

  - To find an instance of the template Box, we have to solve a constraint satisfaction problem of the following form

$$\exists l, u, \forall v: v \in Box(l,u) \Longrightarrow Post_A(v) \in Box(l,u)$$

- **Approach 2: Learning invariants by using execution data**
  - (Si et al, 2018) used RL approach to find invariant from execution data
  - (Kozarev et al, 2016) show how SVM and RL can be used to classify whether states satisfy an invariant based on execution data

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Thank You!!!

Q&A